The Falsifier Calculus: A Novel Approach to Hilbert's Epsilon-Calculus in Deep Inference

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- Recent interest in developing first-order proof systems which admit non-elementarily smaller cut-free proofs than traditional Gentzen systems
- In today's talk:
  - overview deep inference and speedups over traditional proof systems
  - overview Herbrand's Theorem and cut elimination
  - how the epsilon-calculus can help us to understand the non-elementary compression of cut-free proofs and yield new normalisation results

Deep Inference

- More flexible composition mechanism than traditional Gentzen systems using the *open deduction* formalism [Guglielmi, Gundersen, Parigot, 2010]

$$\begin{array}{c} A \\ \phi \Big\| \\ B \end{array} \wedge \begin{array}{c} C \\ \psi \Big\| \\ D \end{array} \equiv \begin{array}{c} A \wedge C \\ \phi\wedge\psi \Big\| \\ B \wedge D \end{array}$$

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\qquad
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$$

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- Applying inference rules at arbitrary depth inside formulae yields improved normalisation properties

$$K \left\{ \rho \frac{A}{B} \right\}$$

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- Can permute inference rules to stratify derivations, revealing *decomposition theorems* for many logics not observed in Gentzen systems

$$\begin{array}{c} \Big\Vert \{\rho_1, \rho_2, \ldots, \rho_n\} \\ A \end{array} \quad \to \quad \begin{array}{c} \Big\Vert \{\rho_1, \ldots, \rho_i\} \\ A' \\ \Big\Vert \{\rho_{i+1}, \ldots, \rho_j\} \\ A'' \\ \Big\Vert \{\rho_{j+1}, \ldots, \rho_n\} \\ A \end{array}$$

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  - useful design principle for proof systems, including *combinatorial proofs* [Hughes, 2006]

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$$

$$
\downarrow
$$

$$
=\,\frac{\forall x(a(x) \wedge b(x)) \vee \forall x(a(x) \wedge b(x))}{\forall x \boxed{\mathsf{m}\,\dfrac{\boxed{\forall\,\dfrac{\forall y(a(y) \wedge b(y))}{a(x) \wedge b(x)}} \vee \boxed{\forall\,\dfrac{\forall y(a(y) \wedge b(y))}{a(x) \wedge b(x)}}}{\boxed{\mathsf{ac}{\downarrow}\,\dfrac{a(x) \vee a(x)}{a(x)}} \wedge \boxed{\mathsf{ac}{\downarrow}\,\dfrac{b(x) \vee b(x)}{b(x)}}}}}
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Deep Inference

- Normalisation results for propositional logic
  - quasipolynomial-complexity cut elimination [Jeřábek, 2008]
  - normalisation procedures which are independent of connective information using *atomic flows* [Gundersen, 2009; Guglielmi, Gundersen, Straßburger, 2010]
  - exponential speedups over cut-free sequent calculus [Bruscoli, Guglielmi, 2009]

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- Can express logics not expressible in the sequent calculus (e.g. self-dual non-commutative binary connectives [Guglielmi, 2007])

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  - exponential speedups over cut-free sequent calculus [Bruscoli, Guglielmi, 2009]
- Can express logics not expressible in the sequent calculus (e.g. self-dual non-commutative binary connectives [Guglielmi, 2007])
- Deep-inference systems do not admit the subformula property and have a larger search space than Gentzen systems for proof search

Proof System

$$
\text{i}\!\downarrow \frac{\mathbf{t}}{A \vee \bar{A}} \qquad \text{w}\!\downarrow \frac{\mathbf{f}}{A} \qquad \text{c}\!\downarrow \frac{A \vee A}{A} \qquad \text{s} \frac{A \wedge (B \vee C)}{(A \wedge B) \vee C} \qquad \exists \frac{A(t)}{\exists x A(x)}
$$

$$
\text{i}\!\uparrow \frac{\bar{A} \wedge A}{\mathbf{f}} \qquad \text{w}\!\uparrow \frac{A}{\mathbf{t}} \qquad \text{c}\!\uparrow \frac{A}{A \wedge A} \qquad \text{m} \frac{(A \wedge B) \vee (C \wedge D)}{(A \vee C) \wedge (B \vee D)} \qquad \forall \frac{\forall x A(x)}{A(t)}
$$

$$
\text{r1}\!\downarrow \frac{\forall x(A(x) \vee B)}{\forall x A(x) \vee B} \qquad \text{r2}\!\downarrow \frac{\forall x(A(x) \wedge B)}{\forall x A(x) \wedge B} \qquad \text{r3}\!\downarrow \frac{\exists x(A(x) \vee B)}{\exists x A(x) \vee B} \qquad \text{r4}\!\downarrow \frac{\exists x(A(x) \wedge B)}{\exists x A(x) \wedge B}
$$

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$$

where $x \notin fv(B)$

+ *equality rules for connective commutativity and associativity, unit equations, quantifier ordering, vacuous quantification and quantifier renaming*

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$+$ *equality rules for connective commutativity and associativity, unit equations, quantifier ordering, vacuous quantification and quantifier renaming*

- *Proofs* are derivations with premise **t** (true)

$$
\begin{array}{c} \mathbf{t} \\ \| \\ A \end{array} \quad \equiv \quad \begin{array}{c} \overline{\|} \\ A \end{array}
$$

Proof System

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+ *equality rules for connective commutativity and associativity, unit equations, quantifier ordering, vacuous quantification and quantifier renaming*

- Every inference rule $\rho{\downarrow}$ has a corresponding *dual* inference rule $\rho{\uparrow}$

$$\rho{\downarrow}\ \frac{A}{B} \qquad \rho{\uparrow}\ \frac{\bar{B}}{\bar{A}}$$

$$\begin{array}{c} A \\ \| \\ B \end{array} \qquad \begin{array}{c} \bar{B} \\ \| \\ \bar{A} \end{array}$$

Quantifier-Shifts

$$\text{r1}{\downarrow}\ \frac{\forall x(A(x) \vee B)}{\forall x A(x) \vee B} \qquad \text{r2}{\downarrow}\ \frac{\forall x(A(x) \wedge B)}{\forall x A(x) \wedge B} \qquad \text{r3}{\downarrow}\ \frac{\exists x(A(x) \vee B)}{\exists x A(x) \vee B} \qquad \text{r4}{\downarrow}\ \frac{\exists x(A(x) \wedge B)}{\exists x A(x) \wedge B}$$

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- **Theorem** [Aguilera, Baaz, 2019]: For a class of tautologies S due to Statman (1979), there is no elementary function bounding the length of the shortest cut-free **LK** proof of any formula in S by its shortest cut-free $\mathbf{LK} + QS$ proof

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- **Theorem** [Aguilera, Baaz, 2019]: For a class of tautologies S due to Statman (1979), there is no elementary function bounding the length of the shortest cut-free **LK** proof of any formula in S by its shortest cut-free **LK** $+ QS$ proof

- **Corollary**: First-order deep-inference systems admit non-elementarily smaller cut-free proofs than **LK**

Herbrand's Theorem

- Existential contraction rules may be understood as case analyses on the existential quantifiers in the premise

$$\mathsf{qc}{\downarrow}\,\frac{\exists x A \vee \exists x A}{\exists x A} \qquad\qquad \mathsf{qc}{\uparrow}\,\frac{\forall x A}{\forall x A \wedge \forall x A}$$

Herbrand's Theorem

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$$\mathsf{qc}{\downarrow}\,\frac{\exists x A \vee \exists x A}{\exists x A} \qquad\qquad \mathsf{qc}{\uparrow}\,\frac{\forall x A}{\forall x A \wedge \forall x A}$$

- A semantically natural operation is thus to extract these case analyses from a proof, deriving a disjunction of terms which witness the existential quantifiers in the conclusion

Herbrand's Theorem

- In a deep-inference setting, Brünnler (2001) has presented a proof of the general version of Herbrand's Theorem in the form of a decomposition theorem

$$
\begin{array}{l}
\Vert \text{First-order rules} \\
A
\end{array}
\quad \rightarrow \quad
\begin{array}{l}
\forall \overrightarrow{x} \boxed{\begin{array}{l} \Vert \text{Propositional rules} \\ A' \end{array}} \\
\qquad \Vert \{\exists\} \\
\qquad A'' \\
\qquad \Vert \{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{r3}\!\downarrow,\mathsf{r4}\!\downarrow\} \\
\qquad A''' \\
\qquad \Vert \{\mathsf{qc}\!\downarrow\} \\
\qquad A
\end{array}
$$

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- $\forall \vec{x} A'$ is called a *Herbrand disjunction for A*

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- $\forall\vec{x}\,A'$ is called a *Herbrand disjunction for A*
- Reduction of undecidable first-order provability to decidable propositional provability

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- $\forall \vec{x}\, A'$ is called a *Herbrand disjunction for A*
- Reduction of undecidable first-order provability to decidable propositional provability
- Non-elementary blowups in proof complexity

Example: The Drinker Paradox

$$
\mathsf{qc}{\downarrow}\,\dfrac{
\begin{array}{c}
\exists\,\dfrac{\exists\,\dfrac{=\dfrac{=\dfrac{\mathbf{t}}{\forall x_1\forall x_2\boxed{=\dfrac{\mathbf{t}}{\boxed{\mathsf{w}{\downarrow}\dfrac{\mathbf{f}}{\overline{P}(c)}}\vee\boxed{\mathsf{i}{\downarrow}\dfrac{\mathbf{t}}{P(x_1)\vee\overline{P}(x_1)}}\vee\boxed{\mathsf{w}{\downarrow}\dfrac{\mathbf{f}}{P(x_2)}}}}}}{\forall x_1\forall x_2((P(x_1)\vee\overline{P}(c))\vee(P(x_2)\vee\overline{P}(x_1)))}}{\forall x_1\exists y_2\forall x_2((P(x_1)\vee\overline{P}(c))\vee(P(x_2)\vee\overline{P}(y_2)))}}{\exists y_1\forall x_1\exists y_2\forall x_2((P(x_1)\vee\overline{P}(y_1))\vee(P(x_2)\vee\overline{P}(y_2)))}\\
\Big\|\,\{\mathsf{r1}{\downarrow},\mathsf{r3}{\downarrow}\}\\
\exists y_1(\forall x_1 P(x_1)\vee\overline{P}(y_1))\vee\exists y_2(\forall x_2 P(x_2)\vee\overline{P}(y_2))
\end{array}}{\exists y(\forall x P(x)\vee\overline{P}(y))}
$$

Herbrand disjunction:

$$
\forall x_1\forall x_2(P(x_1)\vee\overline{P}(c)\vee P(x_2)\vee\overline{P}(x_1))
$$

Cut Elimination

- Elimination of cut rules from a proof

$$\mathrm{i}\!\uparrow \frac{A \wedge \overline{A}}{\mathbf{f}}$$

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- In Gentzen systems, cut rules are permuted up the proof, resulting in non-determinism from certain reduction steps

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- Elimination of cut rules from a proof

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- In Gentzen systems, cut rules are permuted up the proof, resulting in non-determinism from certain reduction steps

- Non-elementary blowups in proof complexity for first-order proofs

Cuts and Quantifier-Shifts

- Cuts on quantifiers are simulated by r1↑ quantifier-shifts and $\forall$ rules

$$\mathrm{i}\!\uparrow\dfrac{\forall xA(x)\wedge\exists x\overline{A}(x)}{\mathbf{f}} \quad\to\quad =\dfrac{{\color{red}\mathrm{r1}\!\uparrow}\dfrac{=\dfrac{\forall xA(x)\wedge\exists x\overline{A}(x)}{\forall yA(y)\wedge\exists x\overline{A}(x)}}{\exists x\boxed{\mathrm{i}\!\uparrow\dfrac{\boxed{{\color{red}\forall}\dfrac{\forall yA(y)}{A(x)}}\wedge\overline{A}(x)}{\mathbf{f}}}}}{\mathbf{f}}$$

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$$
\mathsf{i}{\uparrow}\,\frac{\forall x A(x) \wedge \exists x \bar{A}(x)}{\mathbf{f}}
\quad \to \quad
=\frac{
\color{red}{\mathsf{r1}{\uparrow}}\,\dfrac{
=\dfrac{\forall x A(x) \wedge \exists x \bar{A}(x)}{\forall y A(y) \wedge \exists x \bar{A}(x)}
}{
\exists x \boxed{\mathsf{i}{\uparrow}\,\dfrac{\boxed{\color{red}{\forall}\,\dfrac{\forall y A(y)}{A(x)}} \wedge \bar{A}(x)}{\mathbf{f}}}
}
}{\mathbf{f}}
$$

- Cuts can be reduced to quantifier-shifts and atomic cuts in deep inference

Cut Elimination

- Cut rules may be decomposed to atomic form in deep inference

$$\mathsf{ai}\!\uparrow \frac{a \wedge \bar{a}}{\mathbf{f}}$$

Cut Elimination

- Cut rules may be decomposed to atomic form in deep inference

$$\mathsf{ai}\!\uparrow\frac{a \wedge \bar{a}}{\mathbf{f}}$$

- In propositional deep inference, a semantically natural cut-elimination procedure exists, called the *experiments method* [Guglielmi, 2002; Ralph, 2019]

$$\begin{array}{c}\phi\,\Big\Vert\,\text{Propositional rules}\\ A\end{array} \quad\to\quad \begin{array}{c}\Big\Vert\\ \boxed{\boxed{\begin{array}{c}a\wedge\cdots\wedge a\\ \phi^+\Big\Vert\\ A^+\end{array}} \vee \boxed{\begin{array}{c}\bar{a}\wedge\cdots\wedge\bar{a}\\ \phi^-\Big\Vert\\ A^-\end{array}}}\\ \Big\Vert\\ A\end{array} \qquad \boxed{\begin{array}{l}S^+ = S[\mathbf{f}/\bar{a}]\\ S^- = S[\mathbf{f}/a]\end{array}}$$

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$$\begin{array}{c}\phi\,\Big\Vert\,\text{Propositional rules}\\ A\end{array} \quad\rightarrow\quad \begin{array}{c} \overline{\Big\Vert} \\ \boxed{\boxed{\begin{array}{c} a\wedge\cdots\wedge a \\ \phi^+ \Big\Vert \\ A^+\end{array}} \vee \boxed{\begin{array}{c}\bar a \wedge\cdots\wedge \bar a\\ \phi^-\Big\Vert \\ A^-\end{array}}} \\ \Big\Vert \\ A\end{array} \quad \boxed{\begin{array}{l} S^+ = S[\mathbf{f}/\bar a] \\ S^- = S[\mathbf{f}/a]\end{array}}$$

- Can be extended to first-order logic by first translating to Herbrand normal form

Relationship Between First-Order Phenomena

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Relationship Between First-Order Phenomena

- Traditionally, in the sequent calculus, Herbrand's Theorem is proved as a corollary to cut elimination
- Brünnler's proof of Herbrand's Theorem using deep inference does not require cuts to be eliminated from the proof, establishing a kind of independence between Herbrand's Theorem and cut elimination
- Both cut elimination and Herbrand's Theorem eliminate quantifier-shifts, resulting in non-elementary blowups. Can quantifier-shifts help us to understand the relationship between these theorems?

Elimination of Quantifier-Shifts

- Most quantifier-shifts are derivable with other rules, with the exceptions of r1$\downarrow$ and r1$\uparrow$

$$\text{r2}\!\downarrow \frac{\forall x(A(x) \wedge B)}{\forall x A(x) \wedge B} \quad \rightarrow \quad = \cfrac{\text{qc}\!\uparrow \cfrac{\forall x(A(x) \wedge B)}{\boxed{\forall x \boxed{A(x) \wedge \boxed{\text{w}\!\uparrow \frac{B}{\mathbf{t}}}} \wedge \forall x \boxed{\boxed{\text{w}\!\uparrow \frac{A(x)}{\mathbf{t}}} \wedge B}}}}{\forall x A(x) \wedge B}$$

$$\boxed{\begin{array}{c} \text{r1}\!\downarrow \dfrac{\forall x(A(x) \vee B)}{\forall x A(x) \vee B} \\[2ex] \text{r1}\!\uparrow \dfrac{\exists x A(x) \wedge B}{\exists x(A(x) \wedge B)} \end{array}}$$

$$\text{r3}\!\downarrow \frac{\exists x(A(x) \vee B)}{\exists x A(x) \vee B} \quad \rightarrow \quad = \cfrac{\exists x \boxed{\boxed{\exists \frac{A(x)}{\exists y A(y)}} \vee B}}{\exists x A(x) \vee B}$$

$$\text{r4}\!\downarrow \frac{\exists x(A(x) \wedge B)}{\exists x A(x) \wedge B} \quad \rightarrow \quad = \cfrac{\exists x \boxed{\boxed{\exists \frac{A(x)}{\exists y A(y)}} \wedge B}}{\exists x A(x) \wedge B}$$

Falsifiers

$$\text{r1}\downarrow \frac{\forall x \boxed{A(x) \vee \boxed{\exists \dfrac{B(x)}{\exists y B(y)}}}}{\forall x A(x) \vee \exists y B(y)}$$

- No constructive witness for $\exists y$ given in the derivation

Falsifiers

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\mathsf{r1}{\downarrow}\,\frac{\forall x \boxed{A(x) \vee \boxed{\exists\,\dfrac{B(x)}{\exists y B(y)}}}}{\forall x A(x) \vee \exists y B(y)}
$$

- No constructive witness for $\exists y$ given in the derivation

$$
\exists y = \begin{cases} e & \text{if there exists some } e \in \mathbb{D} \text{ s.t. } \overline{A}(e) \\ a & \text{for some arbitrary } a \in \mathbb{D}\text{, otherwise} \end{cases}
$$

Falsifiers

$$\mathsf{r1}\!\downarrow\ \frac{\forall x \boxed{A(x) \vee \boxed{\exists\ \dfrac{B(x)}{\exists y\, B(y)}}}}{\forall x A(x) \vee \exists y\, B(y)}$$

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$$\begin{aligned}\exists y &= \begin{cases} e & \text{if there exists some } e \in \mathbb{D} \text{ s.t. } \overline{A}(e) \\ a & \text{for some arbitrary } a \in \mathbb{D}\text{, otherwise} \end{cases} \\ &= [\![\varepsilon_x \overline{A}(x)]\!]_{\mathbb{D}} \end{aligned}$$

- Quantifier-shifts implicitly introduce epsilon-terms!

Epsilon-Calculus

- Extend the language of predictate logic by *epsilon-terms*:

$$[\![\varepsilon_x A(x)]\!]_{\mathbb{D}} = \begin{cases} e & \text{if there exists some } e \in \mathbb{D} \text{ s.t. } A(e) \\ a & \text{for some arbitrary } a \in \mathbb{D}\text{, otherwise} \end{cases}$$

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- Introduced by Hilbert (1927) as a tool for developing consistency proofs
- It is known that the traditional epsilon-calculus admits non-elementarily smaller cut-free proofs than **LK** [Baaz, Lolić, 2024] and may be used to obtain speedups in the computation of Herbrand disjunctions over traditional methods [Baaz, Leitsch, Lolić, 2017]

Epsilon-Calculus

- In the traditional epsilon-calculus, epsilon-terms are introduced by *critical axioms* and quantifiers are encoded by epsilon-terms:

$$CA \frac{A(t)}{A(\varepsilon_x A(x))} \qquad \begin{array}{c} \exists x A(x) \equiv A(\varepsilon_x A(x)) \\ \forall x A(x) \equiv A(\varepsilon_x \overline{A}(x)) \end{array}$$

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- Epsilon-terms are eliminated from proofs by "epsilon substitution", which recursively eliminates critical axioms to reduce proofs to propositional form

- From [Baaz, Leitsch, Lolić, 2017]:

> Despite the advantages, the ε-calculus has never become popular in computational proof theory of first order logic. The main reasons are the untractability of almost all nonclassical logics by any adaptation of the ε-formalism and the clumsiness of the ε-formalism itself: consider the ε-translation of $\exists x \exists y \exists z$ $A(x, y, z)$: $A(\varepsilon_x\ A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), \varepsilon_z A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), z)),$ $\varepsilon_y\ A(\varepsilon_x\ A(x, \varepsilon_y\ A(x, y, \varepsilon_z A(x, y, z)), \varepsilon_z A(x, \varepsilon_y A(x, y, \varepsilon_z A(x, y, z)), z)), y, \varepsilon_z A(\varepsilon_x$ $A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), \varepsilon_z\ A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), z)), y, z)),\ \varepsilon_z A(\varepsilon_x$ $A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), \varepsilon_z A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), z)),\ \varepsilon_y A(\varepsilon_x\ A(x, \varepsilon_y$ $A(x, y, \varepsilon_z\ A(x, y, z)), \varepsilon_z\ A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), z)), y, \varepsilon_z\ A(\varepsilon_x\ A(x, \varepsilon_y\ A(x,$ $y,\ \varepsilon_z\ A(x,\ y,\ z)), \varepsilon_z\ A(x, \varepsilon_y\ A(x, y, \varepsilon_z\ A(x, y, z)), z)), y, z)), z)).$

Falsifier Rule

$$\varepsilon \frac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))}$$

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$$\mathsf{r1}{\downarrow}\,\frac{\forall x \boxed{A(x) \vee \boxed{\exists\,\dfrac{B(x)}{\color{blue}{\exists y}\color{black}{B(y)}}}}}{\forall x A(x) \vee \color{red}{\exists y}\color{black}{B(y)}} \quad \to \quad \varepsilon\,\frac{\forall x(A(x) \vee B(x))}{\boxed{\forall x A(x) \vee \boxed{\exists\,\dfrac{B(\varepsilon_y \overline{A}(y))}{\color{red}{\exists y}\color{black}{B(y)}}}}}$$

- Existential rules can be permuted down through quantifier-shifts

Falsifier Rule

$$\varepsilon \, \frac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))}$$

$$\mathrm{r1}{\downarrow} \, \frac{\forall x(A(x) \vee B)}{\forall x A(x) \vee B}$$

- Equivalent to r1↓ when x does not occur free in B

Falsifier Rule

$$\varepsilon\,\frac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))}$$

$$\mathrm{r1}\!\uparrow\,\frac{\boxed{\begin{array}{c} A(t_1) \vee \cdots \vee A(t_n) \\ \Big\Vert\, \{\exists,\mathrm{qc}\!\downarrow\} \\ \exists x A(x) \end{array}} \wedge B}{\exists x(A(x) \wedge B)} \quad \to \quad \begin{array}{c} \boxed{(A(t_1) \vee \cdots \vee A(t_n)) \wedge \boxed{\begin{array}{c} B \\ \Big\Vert\, \{\mathrm{c}\!\uparrow\} \\ B \wedge \cdots \wedge B \end{array}}} \\ \Big\Vert\, \{\mathrm{s}\} \\ \boxed{\begin{array}{c} (A(t_1) \wedge B) \vee \cdots \vee (A(t_n) \wedge B) \\ \Big\Vert\, \{\exists,\mathrm{qc}\!\downarrow\} \\ \exists x(A(x) \wedge B) \end{array}} \end{array}$$

- r1↑ rules may be eliminated using falsifiers – $\exists x A(x)$ may be assigned an explicit disjunction of witnesses $A(t_1) \vee \cdots \vee A(t_n)$

Falsifier Rule

$$\varepsilon \frac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))}$$

- Provides a new perspective on the non-elementary compression yielded by quantifier-shifts
- The epsilon-calculus provides a syntax for expressing non-constructive witnesses to existential quantifiers generated in proofs
- Does not use the cumbersome encodings of quantifiers by epsilon-terms

Extraction of Case Analyses

- A semantically natural operation to perform on a first-order proof is to extract the case analyses contained in quantifier contraction rules

$$\mathsf{qc}\!\downarrow \frac{\exists x A \vee \exists x A}{\exists x A} \qquad\qquad \mathsf{qc}\!\uparrow \frac{\forall x A}{\forall x A \wedge \forall x A}$$

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- Herbrand's Theorem also eliminates quantifier-shifts by making the proof constructive, resulting in non-elementary blowups – can falsifiers prevent this?

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- Herbrand's Theorem also eliminates quantifier-shifts by making the proof constructive, resulting in non-elementary blowups – can falsifiers prevent this?

- Simple case analysis extraction by permuting quantifier contraction rules

Non-Termination of Case Analysis Extraction

$$
\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \vee \exists y B)}{\boxed{\boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall x A}{\forall x A \wedge \forall x A}} \vee \exists y B}}
\quad \rightarrow \quad
\mathsf{2s}\,\dfrac{\mathsf{qc}{\uparrow}\,\dfrac{\forall x(A \vee \exists y B)}{\boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \vee \exists y B)}{\forall x A \vee \exists y B}} \wedge \boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \vee \exists y B)}{\forall x A \vee \exists y B}}}}{\boxed{(\forall x A \wedge \forall x A) \vee \boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists y B \vee \exists y B}{\exists y B}}}}
$$

Non-Termination of Case Analysis Extraction

$$\text{r1}\downarrow\dfrac{\forall x(A \lor \exists yB)}{\boxed{\boxed{\text{qc}\uparrow\dfrac{\forall xA}{\forall xA \land \forall xA}} \lor \exists yB}} \quad \to \quad \text{qc}\uparrow\dfrac{\forall x(A \lor \exists yB)}{\text{2s}\dfrac{\boxed{\text{r1}\downarrow\dfrac{\forall x(A \lor \exists yB)}{\forall xA \lor \exists yB}} \land \boxed{\text{r1}\downarrow\dfrac{\forall x(A \lor \exists yB)}{\forall xA \lor \exists yB}}}{\boxed{(\forall xA \land \forall xA) \lor \boxed{\text{qc}\downarrow\dfrac{\exists yB \lor \exists yB}{\exists yB}}}}}$$

- Superfluous qc$\downarrow$ rules are introduced, despite the witnesses to the existential quantifiers being equal

Non-Termination of Case Analysis Extraction

$$\mathsf{r1}{\downarrow}\,\frac{\forall x(A \lor \exists yB)}{\boxed{\boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall xA}{\forall xA \land \forall xA}} \lor \exists yB}} \quad \to \quad \mathsf{qc}{\uparrow}\,\frac{\forall x(A \lor \exists yB)}{\mathsf{2s}\,\dfrac{\boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \lor \exists yB)}{\forall xA \lor \exists yB}} \land \boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \lor \exists yB)}{\forall xA \lor \exists yB}}}{\boxed{(\forall xA \land \forall xA) \lor \boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists yB \lor \exists yB}{\exists yB}}}}}$$

$$\mathsf{r1}{\uparrow}\,\frac{\boxed{\boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists xA \lor \exists xA}{\exists xA}} \land \forall yB}}{\exists x(A \land \forall yB)} \quad \to \quad \mathsf{qc}{\downarrow}\,\frac{\mathsf{2s}\,\dfrac{\boxed{(\exists xA \lor \exists xA) \land \boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall yB}{\forall yB \land \forall yB}}}}{\boxed{\mathsf{r1}{\uparrow}\,\dfrac{\exists xA \land \forall yB}{\exists x(A \land \forall yB)}} \lor \boxed{\mathsf{r1}{\uparrow}\,\dfrac{\exists xA \land \forall yB}{\exists x(A \land \forall yB)}}}}{\exists x(A \land \forall yB)}$$

- Superfluous qc↓ rules are introduced, despite the witnesses to the existential quantifiers being equal

Non-Termination of Case Analysis Extraction

$$\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \lor \exists yB)}{\boxed{\boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall xA}{\forall xA \land \forall xA}} \lor \exists yB}} \quad \rightarrow \quad \mathsf{2s}\,\dfrac{\mathsf{qc}{\uparrow}\,\dfrac{\forall x(A \lor \exists yB)}{\boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \lor \exists yB)}{\forall xA \lor \exists yB}} \land \boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \lor \exists yB)}{\forall xA \lor \exists yB}}}}{\boxed{(\forall xA \land \forall xA) \lor \boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists yB \lor \exists yB}{\exists yB}}}}$$

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$$\mathsf{r1}{\uparrow}\,\dfrac{\mathsf{m}\,\dfrac{\mathsf{s}\,\dfrac{\boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \lor \exists yB)}{\boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall xA}{\forall xA \land \forall xA}} \lor \exists yB}} \land C}{(\forall xA \land \forall xA) \lor (\exists yB \land C)}}{\boxed{\mathsf{r2}{\uparrow}\,\dfrac{\forall xA \lor \exists yB}{\exists y(\forall xA \lor B)}} \land (\forall xA \lor C)}}{\exists y((\forall xA \lor B) \land (\forall xA \lor C))}$$

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Non-Termination of Case Analysis Extraction

$$\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \vee \exists yB)}{\boxed{\boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall xA}{\forall xA \wedge \forall xA}} \vee \exists yB}} \quad\to\quad \mathsf{qc}{\uparrow}\,\dfrac{\forall x(A \vee \exists yB)}{\mathsf{2s}\,\dfrac{\boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \vee \exists yB)}{\forall xA \vee \exists yB}} \wedge \boxed{\mathsf{r1}{\downarrow}\,\dfrac{\forall x(A \vee \exists yB)}{\forall xA \vee \exists yB}}}{\boxed{(\forall xA \wedge \forall xA) \vee \boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists yB \vee \exists yB}{\exists yB}}}}}$$

$$\mathsf{r1}{\uparrow}\,\dfrac{\boxed{\boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists xA \vee \exists xA}{\exists xA}} \wedge \forall yB}}{\exists x(A \wedge \forall yB)} \quad\to\quad \mathsf{qc}{\downarrow}\,\dfrac{\mathsf{2s}\,\dfrac{\boxed{(\exists xA \vee \exists xA) \wedge \boxed{\mathsf{qc}{\uparrow}\,\dfrac{\forall yB}{\forall yB \wedge \forall yB}}}}{\boxed{\mathsf{r1}{\uparrow}\,\dfrac{\exists xA \wedge \forall yB}{\exists x(A \wedge \forall yB)}} \vee \boxed{\mathsf{r1}{\uparrow}\,\dfrac{\exists xA \wedge \forall yB}{\exists x(A \wedge \forall yB)}}}}{\exists x(A \wedge \forall yB)}$$

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Non-Termination of Case Analysis Extraction

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$$\mathsf{r1}{\uparrow}\,\cfrac{\boxed{\boxed{\mathsf{qc}{\downarrow}\,\cfrac{\exists xA \lor \exists xA}{\exists xA}} \land \forall yB}}{\exists x(A \land \forall yB)} \quad \to \quad \mathsf{qc}{\downarrow}\,\cfrac{\mathsf{2s}\,\cfrac{\boxed{(\exists xA \lor \exists xA) \land \boxed{\mathsf{qc}{\uparrow}\,\cfrac{\forall yB}{\forall yB \land \forall yB}}}}{\boxed{\mathsf{r1}{\uparrow}\,\cfrac{\exists xA \land \forall yB}{\exists x(A \land \forall yB)}} \lor \boxed{\mathsf{r1}{\uparrow}\,\cfrac{\exists xA \land \forall yB}{\exists x(A \land \forall yB)}}}}{\exists x(A \land \forall yB)}$$

$$\mathsf{r1}{\uparrow}\,\cfrac{\mathsf{m}\,\cfrac{\mathsf{s}\,\cfrac{\boxed{\mathsf{r1}{\downarrow}\,\cfrac{\forall x(A \lor \exists yB)}{\boxed{\mathsf{qc}{\uparrow}\,\cfrac{\forall xA}{\forall xA \land \forall xA}} \lor \exists yB}} \land C}{(\forall xA \land \forall xA) \lor (\exists yB \land C)}}{\boxed{\mathsf{r2}{\uparrow}\,\cfrac{\forall xA \lor \exists yB}{\exists y(\forall xA \lor B)}} \land (\forall xA \lor C)}}{\exists y((\forall xA \lor B) \land (\forall xA \lor C))}$$

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Non-Termination of Case Analysis Extraction

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$$\text{r1}\!\uparrow \dfrac{\boxed{\boxed{\text{qc}\!\downarrow \dfrac{\exists xA \lor \exists xA}{\exists xA}} \land \forall yB}}{\exists x(A \land \forall yB)} \quad \to \quad \text{qc}\!\downarrow \dfrac{\text{2s}\,\dfrac{\boxed{(\exists xA \lor \exists xA) \land \boxed{\text{qc}\!\uparrow \dfrac{\forall yB}{\forall yB \land \forall yB}}}}{\boxed{\text{r1}\!\uparrow \dfrac{\exists xA \land \forall yB}{\exists x(A \land \forall yB)}} \lor \boxed{\text{r1}\!\uparrow \dfrac{\exists xA \land \forall yB}{\exists x(A \land \forall yB)}}}}{\exists x(A \land \forall yB)}$$

$$\text{r1}\!\uparrow \dfrac{\text{m}\,\dfrac{\text{s}\,\dfrac{\boxed{\text{r1}\!\downarrow \dfrac{\forall x(A \lor \exists yB)}{\boxed{\text{qc}\!\uparrow \dfrac{\forall xA}{\forall xA \land \forall xA}} \lor \exists yB}} \land C}{(\forall xA \land \forall xA) \lor (\exists yB \land C)}}{\boxed{\text{r2}\!\uparrow \dfrac{\forall xA \lor \exists yB}{\exists y(\forall xA \lor B)}} \land (\forall xA \lor C)}}{\exists y((\forall xA \lor B) \land (\forall xA \lor C))}$$

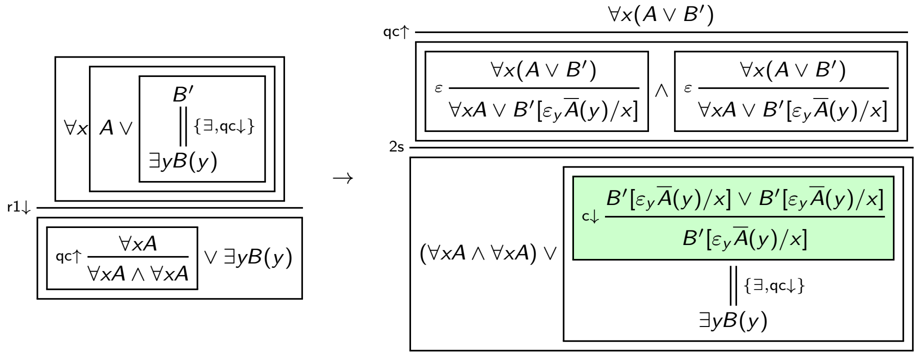
- Superfluous qc$\downarrow$ rules are introduced, despite the witnesses to the existential quantifiers being equal

Termination Using Falsifiers

$$\mathsf{r1}{\downarrow}\ \dfrac{\boxed{\forall x\ \boxed{A \vee \boxed{\begin{array}{c} B' \\ \Big\Vert\ \{\exists,\mathsf{qc}{\downarrow}\} \\ \exists y B(y) \end{array}}}}}{\boxed{\boxed{\mathsf{qc}{\uparrow}\ \dfrac{\forall x A}{\forall x A \wedge \forall x A}} \vee \exists y B(y)}} \quad \to \quad \begin{array}{c} \mathsf{qc}{\uparrow}\ \dfrac{\forall x(A \vee B')}{\mathsf{2s}\ \dfrac{\boxed{\boxed{\varepsilon\ \dfrac{\forall x(A \vee B')}{\forall x A \vee B'[\varepsilon_y \overline{A}(y)/x]}} \wedge \boxed{\varepsilon\ \dfrac{\forall x(A \vee B')}{\forall x A \vee B'[\varepsilon_y \overline{A}(y)/x]}}}}{\boxed{(\forall x A \wedge \forall x A) \vee \boxed{\begin{array}{c} \boxed{\mathsf{c}{\downarrow}\ \dfrac{B'[\varepsilon_y \overline{A}(y)/x] \vee B'[\varepsilon_y \overline{A}(y)/x]}{B'[\varepsilon_y \overline{A}(y)/x]}} \\ \Big\Vert\ \{\exists,\mathsf{qc}{\downarrow}\} \\ \exists y B(y) \end{array}}}}} \end{array}$$

where $B' = B(t_1) \vee \cdots \vee B(t_n)$

Termination Using Falsifiers



where $B' = B(t_1) \vee \cdots \vee B(t_n)$

- The more expressive syntax of the epsilon-calculus can express that the terms are equal – there is no need for a case analysis and so no superfluous $\mathsf{qc}\!\downarrow$ rules are introduced

Extraction of Case Analyses

- Extract case analyses from a first-order proof in three phases:
  - Phase 1: Permute existential contraction rules qc$\downarrow$ down to the bottom of the proof
  - Phase 2: Permute existential instantiation rules $\exists$ down to the bottom of the proof
  - Phase 3: Permute universal cocontraction rules qc$\uparrow$ up the proof until they are eliminated

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  - Phase 1: Permute existential contraction rules $\mathsf{qc}{\downarrow}$ down to the bottom of the proof
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  - Phase 3: Permute universal cocontraction rules $\mathsf{qc}{\uparrow}$ up the proof until they are eliminated

- To begin, eliminate quantifier-shifts as shown

$$\mathsf{r1}{\downarrow}\frac{\forall x(A(x) \vee B)}{\forall x A(x) \vee B} \qquad \rightarrow \qquad \varepsilon\frac{\forall x(A(x) \vee B)}{\forall x A(x) \vee B}$$

Phase 1: Permute qc↓ rules down

- Duplicate rules inside the context of qc↓ rules:

$$\mathsf{qc}{\downarrow}\,\frac{\exists x K\{A\} \vee \exists x K\{A\}}{\boxed{\exists x K\left\{\boxed{\rho\,\frac{A}{B}}\right\}}} \quad \to \quad \mathsf{qc}{\downarrow}\,\frac{\boxed{\boxed{\exists x K\left\{\boxed{\rho\,\frac{A}{B}}\right\}} \vee \boxed{\exists x K\left\{\boxed{\rho\,\frac{A}{B}}\right\}}}}{\exists x K\{B\}}$$

Phase 1: Permute qc↓ rules down

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- Permutation for r1↑ rules:

$$\mathsf{r1}{\uparrow}\,\frac{\boxed{\boxed{\mathsf{qc}{\downarrow}\,\dfrac{\exists x A(x) \vee \exists x A(x)}{\exists x A(x)}} \wedge B}}{\exists x (A(x) \wedge B)} \quad \rightarrow \quad \mathsf{qc}{\downarrow}\,\frac{\mathsf{s}\,\dfrac{\mathsf{s}\,\dfrac{\boxed{(\exists x A(x) \vee \exists x A(x)) \wedge \boxed{\mathsf{c}{\uparrow}\,\dfrac{B}{B \wedge B}}}}{(\exists x A(x) \vee (\exists x A(x) \wedge B)) \wedge B}}{\boxed{\boxed{\mathsf{r1}{\uparrow}\,\dfrac{\exists x A(x) \wedge B}{\exists x (A(x) \wedge B)}} \vee \boxed{\mathsf{r1}{\uparrow}\,\dfrac{\exists x A(x) \wedge B}{\exists x (A(x) \wedge B)}}}}}{\exists x (A(x) \wedge B)}$$

Phase 2: Permute $\exists$ rules down

- Rules inside the context of $\exists$ rules are altered by a substitution:

$$\exists\,\frac{K\{A\}[t/x]}{\boxed{\exists x K\left\{\boxed{\rho\,\dfrac{A}{B}}\right\}}} \quad\to\quad \exists\,\frac{\boxed{K\left\{\boxed{\rho\,\dfrac{A}{B}}\right\}[t/x]}}{\exists x K\{B\}}$$

Phase 2: Permute $\exists$ rules down

- Rules inside the context of $\exists$ rules are altered by a substitution:

$$\exists\,\dfrac{K\{A\}[t/x]}{\boxed{\exists x K\left\{\boxed{\rho\,\dfrac{A}{B}}\right\}}} \quad\to\quad \exists\,\dfrac{\boxed{K\left\{\boxed{\rho\,\dfrac{A}{B}}\right\}[t/x]}}{\exists x K\{B\}}$$

- Permutations for ε rules:

$$\varepsilon\,\dfrac{\boxed{\forall y\left(B(y)\vee K\left\{\boxed{\exists\,\dfrac{A(t)}{\exists x A(x)}}\right\}\right)}}{\forall y B(y)\vee K\{\exists x A\}[\varepsilon_z\overline{B}(z)/y]} \quad\to\quad \varepsilon\,\dfrac{\forall y(B(y)\vee K\{A(t)\})}{\boxed{\forall y B(y)\vee K\left\{\boxed{\exists\,\dfrac{A(t)}{\exists x A(x)}}\right\}[\varepsilon_z\overline{B}(z)/y]}}$$

$$\varepsilon\,\dfrac{\boxed{\forall y\left(K\left\{\boxed{\exists\,\dfrac{A(t)}{\exists x A(x)}}\right\}\vee B(y)\right)}}{\forall y K\{\exists x A(x)\}\vee B(\varepsilon_z(\overline{K\{\exists x A(x)\}[z/y]}))} \quad\to\quad \varepsilon\,\dfrac{\forall y(K\{A(t)\}\vee B(y))}{\boxed{\forall y K\left\{\boxed{\exists\,\dfrac{A(t)}{\exists x A(x)}}\right\}\vee B(\varepsilon_z(\overline{K\{A(t)\}[z/y]}))}}$$

Phase 3: Permute qc↑ rules up

- Duplicate rules inside the context of qc↑ rules:

$$\mathsf{qc}{\uparrow}\,\frac{\boxed{\forall x K\left\{\boxed{\rho\,\dfrac{B}{A}}\right\}}}{\forall x K\{A\} \wedge \forall x K\{A\}} \quad \to \quad \mathsf{qc}{\uparrow}\,\frac{\forall x K\{B\}}{\boxed{\boxed{\forall x K\left\{\boxed{\rho\,\dfrac{B}{A}}\right\}} \wedge \boxed{\forall x K\left\{\boxed{\rho\,\dfrac{B}{A}}\right\}}}}$$

Phase 3: Permute qc↑ rules up

- Duplicate rules inside the context of qc↑ rules:

$$
\dfrac{\boxed{\forall x K\left\{\boxed{\rho\,\dfrac{B}{A}}\right\}}}{\forall x K\{A\} \wedge \forall x K\{A\}}\,\mathsf{qc}\!\uparrow
\quad \to \quad
\dfrac{\forall x K\{B\}}{\boxed{\boxed{\forall x K\left\{\boxed{\rho\,\dfrac{B}{A}}\right\}} \wedge \boxed{\forall x K\left\{\boxed{\rho\,\dfrac{B}{A}}\right\}}}}\,\mathsf{qc}\!\uparrow
$$

- When permuting qc↑ up through ε rules, we employ the following construction, which is invariant under the permutation:

$$
\begin{array}{c}
\boxed{\boxed{\varepsilon\,\dfrac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))}} \wedge \cdots \wedge \boxed{\varepsilon\,\dfrac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))}}} \\
\Big\| \{\mathsf{s}\} \\
\boxed{(\forall x A(x) \wedge \cdots \wedge \forall x A(x)) \vee \boxed{\begin{array}{c} B(\varepsilon_y \overline{A}(y)) \vee \cdots \vee B(\varepsilon_y \overline{A}(y)) \\ \Big\| \{\mathsf{c}\!\downarrow\} \\ B(\varepsilon_y \overline{A}(y)) \end{array}}}
\end{array}
$$

The Falsifier Decomposition Theorem

$$\begin{array}{c} \phi \, \Vert \, \text{First-order rules} \\ A \end{array} \quad \to \quad \begin{array}{c} \phi' \, \Vert \, \text{Propositional rules} + \{\varepsilon, \forall\} \; (\mathsf{SKSg}\varepsilon) \\ A' \\ \Vert \, \{\exists, \mathsf{qc}\!\downarrow\} \\ A \end{array}$$

$$\boxed{\begin{array}{c} \varepsilon \, \dfrac{\forall x (A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))} \\[2ex] \forall \, \dfrac{\forall x A(x)}{A(t)} \end{array}}$$

- I call A' a *falsifier disjunction for A*
- I call SKSgε the *falsifier calculus*

The Falsifier Decomposition Theorem

$$\phi \, \begin{array}{c} \| \\ A \end{array} \text{First-order rules} \quad \rightarrow \quad \begin{array}{c} \phi' \\ \| \; \text{Propositional rules} + \{\varepsilon, \forall\} \; (\mathsf{SKSg}\varepsilon) \\ A' \\ \| \; \{\exists, \mathsf{qc}\!\downarrow\} \\ A \end{array}$$

$$\boxed{\varepsilon \, \frac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))} \qquad \forall \, \frac{\forall x A(x)}{A(t)}}$$

- I call A' a *falsifier disjunction for A*
- I call SKSgε the *falsifier calculus*
- The following bounds hold:

$$|\phi'| = \exp^{10}(O(|\phi|^2 \ln |\phi|))$$
$$|A'| = \exp^{7}(O(|\phi|^2 \ln |\phi|))$$
$$|\phi'|_\varepsilon = \exp^{12}(O(|\phi|^2 \ln |\phi|))$$
$$|A'|_\varepsilon = \exp^{12}(O(|\phi|^2 \ln |\phi|))$$

The Falsifier Decomposition Theorem

$$\begin{array}{c} \phi \\ \Vert \text{First-order rules} \\ A \end{array} \quad \rightarrow \quad \begin{array}{c} \phi' \\ \Vert \text{Propositional rules} + \{\varepsilon, \forall\} \text{ (SKSg}\varepsilon\text{)} \\ A' \\ \Vert \{\exists, \mathsf{qc}\downarrow\} \\ A \end{array}$$

$$\boxed{\begin{array}{c} \varepsilon \dfrac{\forall x(A(x) \vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))} \\[2ex] \forall \dfrac{\forall x A(x)}{A(t)} \end{array}}$$

- I call A' a *falsifier disjunction for A*
- I call SKSgε the *falsifier calculus*
- The following bounds hold:

$$|\phi'| \in \textbf{ELEMENTARY}$$
$$|A'| \in \textbf{ELEMENTARY}$$
$$|\phi'|_\varepsilon \in \textbf{ELEMENTARY}$$
$$|A'|_\varepsilon \in \textbf{ELEMENTARY}$$

- Non-elementarily smaller than Herbrand disjunctions and Herbrand proofs

The Falsifier Decomposition Theorem

$$\begin{array}{c} \phi \\ \Big\Vert \text{ First-order rules} \\ A \end{array} \quad \to \quad \begin{array}{c} \phi' \\ \Big\Vert \text{ Propositional rules} + \{\varepsilon,\forall\} \text{ (SKSg}\varepsilon\text{)} \\ A' \\ \Big\Vert \{\exists,\mathsf{qc}\!\downarrow\} \\ A \end{array}$$

$$\boxed{\begin{array}{c} \varepsilon\,\dfrac{\forall x(A(x)\vee B(x))}{\forall x A(x) \vee B(\varepsilon_y \overline{A}(y))} \\[1em] \forall\,\dfrac{\forall x A(x)}{A(t)} \end{array}}$$

- Unlike Herbrand's Theorem, we have extracted the case analyses from the proof but left the quantifier-shifts intact, in the form of falsifier rules which introduce epsilon-terms
- Epsilon-terms represent elements which are drawn from the domain non-constructively in the proof
- Does not use the cumbersome encodings of quantifiers by epsilon-terms

Example: The Drinker Paradox

$$
=\dfrac{\mathbf{t}}{
=\dfrac{\forall x_1\forall x_2\boxed{=\dfrac{\mathbf{t}}{\boxed{\mathsf{w}{\downarrow}\dfrac{\mathbf{f}}{\overline{P}(c)}}\vee\boxed{\mathsf{i}{\downarrow}\dfrac{\mathbf{t}}{P(x_1)\vee\overline{P}(x_1)}}\vee\boxed{\mathsf{w}{\downarrow}\dfrac{\mathbf{f}}{P(x_2)}}}}}{
\exists\dfrac{\forall x_1\forall x_2((P(x_1)\vee\overline{P}(c))\vee(P(x_2)\vee\overline{P}(x_1)))}{
\exists\dfrac{\forall x_1\exists y_2\forall x_2((P(x_1)\vee\overline{P}(c))\vee(P(x_2)\vee\overline{P}(y_2)))}{\exists y_1\forall x_1\exists y_2\forall x_2((P(x_1)\vee\overline{P}(y_1))\vee(P(x_2)\vee\overline{P}(y_2)))}}}}
$$

$$
\Big\Vert\ \{\mathsf{r1}{\downarrow},\mathsf{r3}{\downarrow}\}
$$

$$
\mathsf{qc}{\downarrow}\dfrac{\exists y_1(\forall x_1 P(x_1)\vee\overline{P}(y_1))\vee\exists y_2(\forall x_2 P(x_2)\vee\overline{P}(y_2))}{\exists y(\forall x P(x)\vee\overline{P}(y))}
$$

$$
=\dfrac{\mathbf{t}}{\varepsilon\dfrac{\forall x\boxed{\mathsf{i}{\downarrow}\dfrac{\mathbf{t}}{P(x)\vee\overline{P}(x)}}}{\exists\dfrac{\forall xP(x)\vee\overline{P}(\varepsilon_y\overline{P}(y))}{\exists y(\forall xP(x)\vee\overline{P}(y))}}}
$$

Herbrand disjunction:

$$\forall x_1\forall x_2(P(x_1)\vee\overline{P}(c)\vee P(x_2)\vee\overline{P}(x_1))$$

Falsifier disjunction:

$$\forall xP(x)\vee\overline{P}(\varepsilon_y\overline{P}(y))$$

Example: The Drinker Paradox

$$
\mathsf{CA}\;\dfrac{\mathrm{i}{\downarrow}\;\dfrac{\mathbf{t}}{P(\varepsilon_x \overline{P}(x)) \vee \overline{P}(\varepsilon_x \overline{P}(x))}}{P(\varepsilon_x \overline{P}(x)) \vee \overline{P}(\varepsilon_y(P(\varepsilon_x \overline{P}(x)) \vee \overline{P}(y)))}
$$

$$
\exists\;\dfrac{\varepsilon\;\dfrac{=\;\dfrac{\mathbf{t}}{\forall x \boxed{\mathrm{i}{\downarrow}\;\dfrac{\mathbf{t}}{P(x) \vee \overline{P}(x)}}}}{\forall x P(x) \vee \overline{P}(\varepsilon_y \overline{P}(y))}}{\exists y(\forall x P(x) \vee \overline{P}(y))}
$$

$$
\boxed{\mathsf{CA}\;\dfrac{A(t)}{A(\varepsilon_x A(x))}}
\qquad
\boxed{\begin{array}{l} \exists x A(x) \equiv A(\varepsilon_x A(x)) \\ \forall x A(x) \equiv A(\varepsilon_x \overline{A}(x)) \end{array}}
$$

Ongoing Work: Herbrand's Theorem

$$
\begin{array}{l}
\Vert \text{Propositional rules} + \{\varepsilon, \forall\} \text{ (SKSg}\varepsilon\text{)} \\
A' \\
\Vert \{\exists, \mathsf{qc}\!\downarrow\} \\
A
\end{array}
\quad \stackrel{?}{\rightarrow} \quad
\begin{array}{l}
\forall \vec{x} \boxed{\begin{array}{l} \Vert \text{Propositional rules} \\ A' \end{array}} \\
\Vert \{\exists\} \\
A'' \\
\Vert \{\mathsf{r1}\!\downarrow, \mathsf{r2}\!\downarrow, \mathsf{r3}\!\downarrow, \mathsf{r4}\!\downarrow\} \\
A''' \\
\Vert \{\mathsf{qc}\!\downarrow\} \\
A
\end{array}
$$

- Given a proof in falsifier normal form, how can we extract a proof in Herbrand normal form?

Ongoing Work: Herbrand's Theorem

$$
\begin{array}{l}
\Vert\ \text{Propositional rules} + \{\varepsilon, \forall\}\ (\mathsf{SKSg}\varepsilon) \\
A' \\
\Vert\ \{\exists,\mathsf{qc}{\downarrow}\} \\
A
\end{array}
\quad \xrightarrow{?} \quad
\begin{array}{l}
\forall \vec{x}\ \boxed{\begin{array}{l} \Vert\ \text{Propositional rules} \\ A' \end{array}} \\
\qquad \Vert\ \{\exists\} \\
\qquad A'' \\
\qquad \Vert\ \{\mathsf{r1}{\downarrow},\mathsf{r2}{\downarrow},\mathsf{r3}{\downarrow},\mathsf{r4}{\downarrow}\} \\
\qquad A''' \\
\qquad \Vert\ \{\mathsf{qc}{\downarrow}\} \\
\qquad A
\end{array}
$$

- Given a proof in falsifier normal form, how can we extract a proof in Herbrand normal form?
- Can provide insight into the relationship between falsifier normal form and other first-order proof interpretations, such as the no-counterexample interpretation

Ongoing Work: Herbrand's Theorem

- Example: the drinker paradox

$$
{\scriptstyle \exists}\,\dfrac{{\scriptstyle \varepsilon}\,\dfrac{{\scriptstyle =}\,\dfrac{\mathbf{t}}{\forall x \boxed{{\scriptstyle \mathrm{i}\downarrow}\,\dfrac{\mathbf{t}}{P(x) \vee \overline{P}(x)}}}}{\forall x P(x) \vee \overline{P}(\varepsilon_x \overline{P}(x))}}{\exists y(\forall x P(x) \vee \overline{P}(y))}
$$

Ongoing Work: Herbrand's Theorem

- Example: the drinker paradox

$$\exists\,\dfrac{\varepsilon\,\dfrac{=\,\dfrac{\mathbf{t}}{\forall x\,\boxed{\mathrm{i}{\downarrow}\,\dfrac{\mathbf{t}}{P(x)\vee\overline{P}(x)}}}}{\forall x P(x)\vee\overline{P}(\varepsilon_x\overline{P}(x))}}{\exists y(\forall x P(x)\vee\overline{P}(y))}$$

- Falsifier disjunction:

$$\forall \boldsymbol{x} P(x)\vee\overline{P}(\boldsymbol{\varepsilon_x}\overline{P}(x))$$

Ongoing Work: Herbrand's Theorem

- Example: the drinker paradox

$$
{\scriptstyle \exists}\,\dfrac{{\scriptstyle \varepsilon}\,\dfrac{{\scriptstyle =}\,\dfrac{\mathbf{t}}{\forall x\,\boxed{\mathrm{i}{\downarrow}\,\dfrac{\mathbf{t}}{P(x)\vee\overline{P}(x)}}}}{\forall x P(x)\vee\overline{P}(\varepsilon_x\overline{P}(x))}}{\exists y(\forall x P(x)\vee\overline{P}(y))}
$$

- Falsifier disjunction:

$$\forall \boldsymbol{x} P(x)\vee\overline{P}(\boldsymbol{\varepsilon_x}\overline{P}(x))$$

- $\vee$-expand $\exists y$:

$$\forall x_1\forall x_2[(P(x_1)\vee\overline{P}(c))\vee(P(x_2)\vee\overline{P}(x_1))]$$

Ongoing Work: Herbrand's Theorem

- Example

$$
\exists\,\dfrac{\exists\,\dfrac{=\,\dfrac{\mathbf{t}}{\boxed{\forall x \forall y\ \boxed{\boxed{\varepsilon\,\dfrac{\boxed{\forall x'\ \boxed{\mathsf{i}{\downarrow}\,\dfrac{\mathbf{t}}{P(x')\vee\overline{P}(x')}}}}{\boxed{\forall\,\dfrac{\forall x' P(x')}{P(x)}}\vee \overline{P}(\varepsilon_x\overline{P}(x))}} \wedge \boxed{\varepsilon\,\dfrac{\boxed{\forall y'\ \boxed{\mathsf{i}{\downarrow}\,\dfrac{\mathbf{t}}{Q(y')\vee\overline{Q}(y')}}}}{\boxed{\forall\,\dfrac{\forall y' Q(y')}{Q(y)}}\vee \overline{Q}(\varepsilon_y\overline{Q}(y))}}}}}}{\exists v \forall x \forall y[(P(x)\vee\overline{P}(\varepsilon_x\overline{P}(x)))\wedge(Q(y)\vee\overline{Q}(v))]}}{\exists u \exists v \forall x \forall y[(P(x)\vee\overline{P}(u))\wedge(Q(y)\vee\overline{Q}(v))]}
$$

Ongoing Work: Herbrand's Theorem

- Example

$$
\exists\,\dfrac{\exists\,\dfrac{=\,\dfrac{\mathbf{t}}{\boxed{\forall x\forall y\;\boxed{\boxed{\varepsilon\,\dfrac{\boxed{\forall x'\;\boxed{\mathrm{i}{\downarrow}\,\dfrac{\mathbf{t}}{P(x')\vee\overline{P}(x')}}}}{\boxed{\forall\,\dfrac{\forall x' P(x')}{P(x)}}\vee\overline{P}(\varepsilon_x\overline{P}(x))}}\;\wedge\;\boxed{\varepsilon\,\dfrac{\boxed{\forall y'\;\boxed{\mathrm{i}{\downarrow}\,\dfrac{\mathbf{t}}{Q(y')\vee\overline{Q}(y')}}}}{\boxed{\forall\,\dfrac{\forall y' Q(y')}{Q(y)}}\vee\overline{Q}(\varepsilon_y\overline{Q}(y))}}}}}}{\exists v\forall x\forall y[(P(x)\vee\overline{P}(\varepsilon_x\overline{P}(x)))\wedge(Q(y)\vee\overline{Q}(v))]}}{\exists u\exists v\forall x\forall y[(P(x)\vee\overline{P}(u))\wedge(Q(y)\vee\overline{Q}(v))]}
$$

- Falsifier disjunction:

$$
\forall \boldsymbol{x}\forall \boldsymbol{y}[(P(x)\vee\overline{P}(\boldsymbol{\varepsilon_x}\overline{P}(x)))\wedge(Q(y)\vee\overline{Q}(\boldsymbol{\varepsilon_y}\overline{Q}(y)))]
$$

Ongoing Work: Herbrand's Theorem

- Formula:

$$\exists u \exists v \forall x \forall y [(P(x) \vee \overline{P}(u)) \wedge (Q(y) \vee \overline{Q}(v))]$$

- Falsifier disjunction:

$$\forall \boldsymbol{x} \forall \boldsymbol{y} [(P(x) \vee \overline{P}(\varepsilon_{\boldsymbol{x}} \overline{P}(x))) \wedge (Q(y) \vee \overline{Q}(\varepsilon_{\boldsymbol{y}} \overline{Q}(y)))]$$

Ongoing Work: Herbrand's Theorem

- Formula:

$$\exists u \exists v \forall x \forall y [(P(x) \vee \overline{P}(u)) \wedge (Q(y) \vee \overline{Q}(v))]$$

- Falsifier disjunction:

$$\forall \boldsymbol{x} \forall \boldsymbol{y} [(P(x) \vee \overline{P}(\varepsilon_{\boldsymbol{x}} \overline{P}(x))) \wedge (Q(y) \vee \overline{Q}(\varepsilon_{\boldsymbol{y}} \overline{Q}(y)))]$$

- $\vee$-expand $\exists u$:

$$\forall x_1 \forall \boldsymbol{y_1} \forall x_2 \forall \boldsymbol{y_2} [((P(x_1) \vee \overline{P}(c)) \wedge (Q(y_1) \vee \overline{Q}(\varepsilon_{\boldsymbol{y_1}} \overline{Q}(y_1)))) \vee \\ ((P(x_2) \vee \overline{P}(x_1)) \wedge (Q(y_2) \vee \overline{Q}(\varepsilon_{\boldsymbol{y_2}} \overline{Q}(y_2))))]$$

Ongoing Work: Herbrand's Theorem

- Formula:

$$\exists u \exists v \forall x \forall y[(P(x) \vee \overline{P}(u)) \wedge (Q(y) \vee \overline{Q}(v))]$$

- Falsifier disjunction:

$$\forall \boldsymbol{x} \forall \boldsymbol{y}[(P(x) \vee \overline{P}(\varepsilon_{\boldsymbol{x}} \overline{P}(x))) \wedge (Q(y) \vee \overline{Q}(\varepsilon_{\boldsymbol{y}} \overline{Q}(y)))]$$

- $\vee$-expand $\exists u$:

$$\begin{aligned}\forall x_1 \forall \boldsymbol{y_1} \forall x_2 \forall \boldsymbol{y_2}[((P(x_1) \vee \overline{P}(c)) \wedge (Q(y_1) \vee \overline{Q}(\varepsilon_{\boldsymbol{y_1}} \overline{Q}(y_1))))\vee \\ ((P(x_2) \vee \overline{P}(x_1)) \wedge (Q(y_2) \vee \overline{Q}(\varepsilon_{\boldsymbol{y_2}} \overline{Q}(y_2))))]\end{aligned}$$

- $\vee$-expand $\exists v$:

$$\begin{aligned}&\forall x_{11} \forall y_{11} \forall x_{12} \forall y_{12} \forall x_{211} \forall y_{211} \forall x_{212} \forall y_{212} \forall x_{221} \forall y_{221} \forall x_{222} \forall y_{222}[\\ &((P(x_{11}) \vee \overline{P}(c)) \wedge (Q(y_{11}) \vee \overline{Q}(c)))\vee \\ &((P(x_{12}) \vee \overline{P}(c)) \wedge (Q(y_{12}) \vee \overline{Q}(y_{11})))\vee \\ &((P(x_{211}) \vee \overline{P}(x_{11})) \wedge (Q(y_{211}) \vee \overline{Q}(c)))\vee \\ &((P(x_{212}) \vee \overline{P}(x_{11})) \wedge (Q(y_{212}) \vee \overline{Q}(y_{211})))\vee \\ &((P(x_{221}) \vee \overline{P}(x_{12})) \wedge (Q(y_{221}) \vee \overline{Q}(c)))\vee \\ &((P(x_{222}) \vee \overline{P}(x_{12})) \wedge (Q(y_{222}) \vee \overline{Q}(y_{221})))]\end{aligned}$$

Ongoing Work: Herbrand's Theorem

$$\begin{array}{c} \Big\Vert \text{Propositional rules} + \{\varepsilon, \forall\} \text{ (SKSg}\varepsilon) \\ A' \\ \Big\Vert \{\exists, \mathsf{qc}\downarrow\} \\ A \end{array} \quad \rightarrow \quad \begin{array}{c} \forall \vec{x} \boxed{\begin{array}{c} \Big\Vert \text{Propositional rules} \\ A' \end{array}} \\ \Big\Vert \{\exists\} \\ A'' \\ \Big\Vert \{\mathsf{r1}\downarrow, \mathsf{r2}\downarrow, \mathsf{r3}\downarrow, \mathsf{r4}\downarrow\} \\ A''' \\ \Big\Vert \{\mathsf{qc}\downarrow\} \\ A \end{array}$$

- General idea: eliminate epsilon-terms by replacing them with Herbrand instances of the drinker paradox

Ongoing Work: Herbrand's Theorem

$$
\begin{array}{l}
\Big\Vert \text{Propositional rules} + \{\varepsilon,\forall\}\ (\mathsf{SKSg}\varepsilon) \\
A' \\
\Big\Vert \{\exists,\mathsf{qc}\!\downarrow\} \\
A
\end{array}
\quad \rightarrow \quad
\begin{array}{l}
\forall\vec{x}\ \boxed{\begin{array}{l} \Big\Vert \text{Propositional rules} \\ A' \end{array}} \\
\Big\Vert \{\exists\} \\
A'' \\
\Big\Vert \{\mathsf{r1}\!\downarrow,\mathsf{r2}\!\downarrow,\mathsf{r3}\!\downarrow,\mathsf{r4}\!\downarrow\} \\
A''' \\
\Big\Vert \{\mathsf{qc}\!\downarrow\} \\
A
\end{array}
$$

- General idea: eliminate epsilon-terms by replacing them with Herbrand instances of the drinker paradox

- Non-elementary blowups due to dependency structure between epsilon-terms

The Logic of Quantifier-Shifts (Joint Work With Raheleh Jalali)

- The logic of quantifier-shifts (QFS) is obtained by extending intuitionistic predicate logic by the quantifier-shift principles and is the smallest intermediate logic to admit Skolemisation [Baaz, Gamsakhurdia, Iemhoff, Jalali, 2025]

The Logic of Quantifier-Shifts (Joint Work With Raheleh Jalali)

- The logic of quantifier-shifts (QFS) is obtained by extending intuitionistic predicate logic by the quantifier-shift principles and is the smallest intermediate logic to admit Skolemisation [Baaz, Gamsakhurdia, Iemhoff, Jalali, 2025]
- The quantifier-shifts which are intuitionistically unsound are equivalent to critical formulae for ε- and τ-terms:

$$\exists x A(x) \to A(\varepsilon_x A(x))$$

$$A(\tau_x A(x)) \to \forall x A(x)$$

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- The quantifier-shifts which are intuitionistically unsound are equivalent to critical formulae for ε- and τ-terms:

$$\exists x A(x) \to A(\varepsilon_x A(x))$$

$$A(\tau_x A(x)) \to \forall x A(x)$$

- We are investigating the relationship between QFS and the intuitionistic epsilon/tau-calculus to better understand the semantics of QFS and develop a novel semantics for the intuitionistic epsilon/tau-calculus

Conclusion

- A new approach to the epsilon-calculus, guided by considerations of complexity and normalisation

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Conclusion

- A new approach to the epsilon-calculus, guided by considerations of complexity and normalisation
- A new decomposition theorem for first-order proofs, which does not fully separate the first-order and propositional parts
- Extending the language of predicate logic by epsilon-terms yields:
  - improved normalisation properties of quantifier-shifts
  - termination of case analysis extraction
  - new perspective on the speedups yielded by quantifier-shifts
  - syntax for expressing non-constructive behaviour of existential witnesses